

# LOCAL SOLVABILITY OF THE $k$ -HESSIAN EQUATIONS

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**ABSTRACT.** In this work, we study the existence of local solutions in  $\mathbb{R}^n$  to  $k$ -Hessian equation, for which the nonhomogeneous term  $f$  is permitted to change the sign or be non negative; if  $f$  is  $C^\infty$ , so is the local solution. We also give a classification for the second order polynomial solutions to the  $k$ -Hessian equation, it is the basis to construct the local solutions and obtain the uniform ellipticity of the linearized operators at such constructed local solutions.

## 1. INTRODUCTION

In this paper, we focus on the existence of local solution for the following  $k$ -Hessian equation,

$$(1.1) \quad S_k[u] = f(y, u, Du),$$

on an open domain  $\Omega$  of  $\mathbb{R}^n$ , where  $2 \leq k \leq n$ . For a smooth function  $u \in C^2$ , the  $k$ -Hessian operator  $S_k$  is defined by

$$(1.2) \quad S_k[u] = S_k(D^2u) = \sigma_k(\lambda(D^2u)) = \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} \lambda_{i_1} \lambda_{i_2} \dots \lambda_{i_k},$$

where  $\lambda(D^2u) = (\lambda_1, \dots, \lambda_n)$  are the eigenvalues of the Hessian matrix  $(D^2u)$ ,  $\sigma_k(\lambda)$  is the  $k$ -th elementary symmetric polynomial, and  $S_k[u]$  is the sum of all principal minors of order  $k$  for the Hessian matrix  $(D^2u)$ . We say that a smooth function  $u$  is  $k$ -convex if the eigenvalues of the Hessian matrix  $(D^2u)$  are in the so-called Gårding cone  $\Gamma_k$  which is defined by

$$\Gamma_k(n) = \{\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbb{R}^n : \sigma_j(\lambda) > 0, 1 \leq j \leq k\}.$$

For the local solvability, Hong and Zuily [7] obtained the existence of  $C^\infty$  local solutions for Monge-Ampère equation

$$(1.3) \quad \det D^2u = f(y, u, Du) \quad y \in \Omega \subset \mathbb{R}^n,$$

when  $f \in C^\infty$  is nonnegative, it is the case of  $k = n$  in (1.1). The geometric background of Monge-Ampère equation can be found in [6, 4, 12]. In this work, we only consider the Hessian equation for  $2 \leq k < n$ , since it is classical for  $k=1$ . We will follow the method of [7] (see also [11, 14]) to construct the local solution by a perturbation of the polynomial-typed solution of  $S_k[u] = c$  for some real constant  $c$ . Since the right hand side function in (1.1) possibly vanishes, then, the solution is in the closure of  $\Gamma_k$ . Thus, we need to study the closure of  $\Gamma_k$ , its boundary is

$$\partial\Gamma_k(n) = \{\lambda \in \mathbb{R}^n : \sigma_j(\lambda) \geq 0, \sigma_k(\lambda) = 0, 1 \leq j \leq k-1\}.$$

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From the Maclaurin's inequalities

$$\left[ \frac{\sigma_k(\lambda)}{\binom{n}{k}} \right]^{1/k} \leq \left[ \frac{\sigma_l(\lambda)}{\binom{n}{l}} \right]^{1/l}, \quad \lambda \in \Gamma_k, k \geq l \geq 1,$$

we see that  $\sigma_{k+1}(\lambda) > 0$  cannot occur for  $\lambda \in \partial\Gamma_k(n)$ , therefore we can express  $\partial\Gamma_k$  as two parts

$$\partial\Gamma_k(n) = \mathbf{P}_1 \cup \mathbf{P}_2,$$

with

$$\begin{aligned} \mathbf{P}_1 &= \{\lambda \in \Gamma_k(n) : \sigma_j(\lambda) \geq 0, \sigma_k(\lambda) = \sigma_{k+1}(\lambda) = 0, 1 \leq j \leq k-1\} \\ \mathbf{P}_2 &= \{\lambda \in \Gamma_k(n) : \sigma_j(\lambda) \geq 0, \sigma_k(\lambda) = 0, \sigma_{k+1}(\lambda) < 0, 1 \leq j \leq k-1\}. \end{aligned}$$

Besides,

$$(1.4) \quad \mathbf{P}_2 = \emptyset, \text{ if } k = n.$$

In Section 2, we will prove that  $\mathbf{P}_1$  and  $\mathbf{P}_2$  have a more precise version, and  $\mathbf{P}_2 \neq \emptyset$  if  $k < n$ .

In this paper, we always discuss the  $k$ -Hessian equation under the framework of ellipticity, then we follow the ideas of [8] and [9] to get the existence of the solution. Here we explain the ellipticity: in the matrix language, the ellipticity set of the  $k$ -Hessian operator,  $1 \leq k \leq n$ , is

$$E_k = \{S \in \mathcal{M}_s(n) : S_k(S + t\xi \times \xi) > S_k(S) > 0, \xi \in \mathbb{S}^{n-1}, \forall t \in \mathbb{R}^+\},$$

where  $\mathcal{M}_s(n)$  is the space of  $n$ -symmetric real matrix. Then the Gårding cones is

$$\Gamma_k = \{S \in \mathcal{M}_s(n) : S_k(S + t\mathbf{I}) > S_k(S) > 0, \forall t \in \mathbb{R}^+\}.$$

It is possible to show that  $E_k = \Gamma_k$  only for  $k = 1, n$  and the example in [9] assures that  $\Gamma_k \subset E_k$  and  $\text{mess}(E_k \setminus \Gamma_k) > 0$ . Ivochkina, Prokofeva and Yakunina [9] pointed out that the ellipticity of (1.1) is independent of the sign of  $f$  if  $k < n$ . But for the Monge-Ampère equation (1.3), the type of equation is determined by the sign of  $f$ , it is elliptic if  $f > 0$ , hyperbolic if  $f < 0$  and degenerate if  $f$  vanishes at some points; it is of mixed type if  $f$  changes sign [5].

There are many results for the Dirichlet problem of (1.1) under the condition  $f > 0$  (see [15] and references therein), there are also some results about  $C^{1,1}$  weak solution of the Dirichlet problem of (1.1) under the degenerate condition  $f \geq 0$  (see [16] and references therein). But similarly to Monge-Ampère equation, the existence of the smooth solution to Dirichlet problem of (1.1) is completely an open problem if  $f$  is not strictly positive.

In this work, for the local solution of the  $k$ -Hessian equation (1.1), we prove the following results.

**Theorem 1.1.** *Let  $f = f(y, u, p)$  be defined and continuous near a point  $Z_0 = (y_0, u_0, p_0) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$ ,  $0 < \alpha < 1, 2 \leq k < n$ . Assume that  $f$  is  $C^\alpha$  with respect to  $y$  and  $C^{2,1}$  with respect to  $u, p$ . We have that*

- (1) *If  $f(Z_0) = 0$ , then the equation (1.1) admits a  $(k-1)$ -convex local solution  $u \in C^{2,\alpha}$  near  $y_0$  which is not  $(k+1)$ -convex.*
- (2) *If  $f \geq 0$  near  $Z_0$ , then the equation (1.1) admits a  $k$ -convex local solution  $u \in C^{2,\alpha}$  near  $y_0$  which is not  $(k+1)$ -convex.*
- (3) *If  $f(Z_0) < 0$ , the equation (1.1) admits a  $(k-1)$ -convex local solution  $u \in C^{2,\alpha}$  near  $y_0$  which must not be  $k$ -convex.*

*Moreover, in all the case above, the linearized operator of (1.1) at  $u$  is uniformly elliptic, and if  $f \in C^\infty$  near  $Z_0$ , then the local solution above is  $C^\infty$  near  $y_0$ .*

**Remark 1.2.** 1) Without loss of generality, by a translation  $y \rightarrow y - y_0$  and replacing  $u$  by  $u - u(0) - y \cdot Du(0)$ , we can assume  $Z_0 = (0, 0, 0)$  in Theorem 1.1, then the local solution in the above Theorem 1.1 is of the following form

$$(1.5) \quad u(y) = \frac{1}{2} \sum_{i=1}^n \tau_i y_i^2 + \varepsilon' \varepsilon^4 w(\varepsilon^{-2} y),$$

with arbitrarily fixed  $(\tau_1, \tau_2, \dots, \tau_n) \in \mathbf{P}_2$  in the cases of (1) and (2). In the case of (3), we take some special  $(\tau_1, \tau_2, \dots, \tau_n) \in \Gamma_{k-1}$ . In (1.5), we always take  $\varepsilon > 0$  and

$$(1.6) \quad \varepsilon' = \begin{cases} \varepsilon^\alpha, & 0 < \alpha \leq \frac{1}{2} \\ \varepsilon, & \frac{1}{2} < \alpha < 1, \end{cases}$$

then (1.5) is of the same form as the solution in [7, 11, 14], where  $f$  has good smoothness.

2) Notice that, in Case (1) of Theorem 1.1,  $f$  is permitted to change sign near  $Z_0$ .

3) If  $f = f(y, u, p)$  is independent of  $u$  and  $p$ , then the assumption on  $f$  is reduced to  $f = f(y) \in C^\alpha$ , which is a necessary requirement on  $f$  for the classical Schauder theory. The condition that  $f$  is  $C^{2,1}$  with respect to  $u$  and  $p$  is a technical one to meet the need for tackling with the quadratic error in Nash-Moser iteration (see (3.14)).

4) In Theorem 1.1, we consider only the  $k$ -Hessian equation with  $2 \leq k < n$ , since the Monge-Ampère equation (1.3) is considered by [5, 7].

This article consists of three sections besides the introduction. In Section 2, we give a classification of the polynomial-typed solutions for  $S_k[u] = c$  for some real constant  $c$ . Such a classification will assure the ellipticity of linearized operators at each polynomial. The results of this section given also a good understanding for the structure of solutions to  $k$ -hessian equation. In Section 3, Theorem 1.1 is proved by Nash-Moser iteration. Section 4 is an appendix in which three equivalent definitions for Gårding cone are given and proved.

## 2. A CLASSIFICATION OF POLYNOMIAL SOLUTIONS

For  $\lambda \in \mathbb{R}^n$ , set  $\psi(y) = \frac{1}{2} \sum_{i=1}^n \lambda_i y_i^2$ , then

$$(2.1) \quad S_k[\psi] = \sigma_k(\lambda) = c,$$

where  $c$  is a real constant. The linearized operators of  $S_k[\cdot]$  at  $\psi$  is

$$(2.2) \quad \mathcal{L}_\psi = \sum_{i=1}^n \sigma_{k-1,i}(\lambda) \partial_i^2,$$

where  $\sigma_{k-1,i}(\lambda)$ , furthermore,  $\sigma_{l,i_1,i_2,\dots,i_s}(\lambda)$ , is defined in (4.7). To give a classification of polynomial solutions to equation (2.1), we recall a results of Section 2 of [15].

**Proposition 2.1.** (See [15]) Assume that  $\lambda \in \bar{\Gamma}_k(n)$  is in descending order,

(i) then we have

$$\lambda_1 \geq \dots \lambda_k \geq \dots \lambda_p \geq 0 \geq \lambda_{p+1} \geq \dots \lambda_n$$

with  $p \geq k$ .

(ii) we have

$$(2.3) \quad 0 \leq \sigma_{k-1,1}(\lambda) \leq \sigma_{k-1,2}(\lambda) \leq \dots \leq \sigma_{k-1,n}(\lambda).$$

(iii) For any  $\{i_1, i_2, \dots, i_s\} \subset \{1, 2, \dots, n\}$  with  $l + s \leq k$ , we have

$$(2.4) \quad \sigma_{l,i_1,i_2,\dots,i_s}(\lambda) \geq 0.$$

Using (ii) of the Proposition 2.1, for any  $\lambda \in \bar{\Gamma}_k(n)$ , the linearized operators  $\mathcal{L}_\psi$  defined in (2.2) could be degenerate elliptic. Now we study the non-strictly  $k$ -convex Garding's cone  $\bar{\Gamma}_k(n)$ , we will show some special uniformly elliptic case.

**Theorem 2.2.** *Suppose that  $\lambda \in \partial\Gamma_k(n) = \mathbf{P}_1 \cup \mathbf{P}_2$ ,  $2 \leq k \leq n-1$ . Then either*

(I) *If  $\sigma_k(\lambda) = 0$  and  $\sigma_{k+1}(\lambda) < 0$ , then*

$$\sigma_{k-1;i}(\lambda) > 0, \quad i = 1, 2, \dots, n.$$

or

(II) *If  $\sigma_k(\lambda) = 0 = \sigma_{k+1}(\lambda)$ , then*

$$\sigma_j(\lambda) = 0, \quad j = k+2, \dots, n,$$

*that means  $\lambda \in \bar{\Gamma}_n(n)$ .*

In order to prove this theorem, we need several lemmas.

**Lemma 2.3.** *Let  $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$  and it is in the descending order. For  $0 \leq s < n-k-1$ , denote  $\lambda^{(s)} = (\lambda_{s+1}, \dots, \lambda_n)$ . Suppose that  $\lambda^{(s)} \in \bar{\Gamma}_k(n-s)$  and*

$$\begin{cases} \sigma_{k-1;s+1}(\lambda^{(s)}) = 0, \\ \sigma_k(\lambda^{(s)}) = 0, \\ \sigma_{k+1}(\lambda^{(s)}) < 0. \end{cases}$$

*Then  $\lambda^{(s+1)} = (\lambda_{s+2}, \dots, \lambda_n) \in \bar{\Gamma}_k(n-s-1)$  and*

$$\begin{cases} \sigma_{k-1;s+2}(\lambda^{(s+1)}) = 0, \\ \sigma_k(\lambda^{(s+1)}) = 0, \\ \sigma_{k+1}(\lambda^{(s+1)}) < 0. \end{cases}$$

*Proof.* It suffices to prove this lemma for  $s = 0$  since we can complete the proof by an induction on the length of  $\lambda^{(s)}$ . Here we call the length of  $\lambda^{(n-j)}$  is  $j$  if  $\lambda^{(n-j)} = (\lambda_{n-j+1}, \dots, \lambda_n)$ . Thus, we suppose that

$$(2.5) \quad \sigma_{k-1;1}(\lambda) = 0, \quad \sigma_k(\lambda) = 0, \quad \sigma_{k+1}(\lambda) < 0.$$

Using (4.8)

$$\sigma_k(\lambda) = \lambda_1 \sigma_{k-1;1}(\lambda) + \sigma_{k;1}(\lambda),$$

and

$$\sigma_{k+1}(\lambda) = \lambda_1 \sigma_{k;1}(\lambda) + \sigma_{k+1;1}(\lambda),$$

we get

$$(2.6) \quad \begin{cases} \sigma_{k-1;1}(\lambda_1, \dots, \lambda_n) = \sigma_{k-1}(\lambda_2, \dots, \lambda_n) = 0, \\ \sigma_{k;1}(\lambda_1, \dots, \lambda_n) = \sigma_k(\lambda_2, \dots, \lambda_n) = 0, \\ \sigma_{k+1;1}(\lambda_1, \dots, \lambda_n) = \sigma_{k+1}(\lambda_2, \dots, \lambda_n) < 0. \end{cases}$$

By (2.4) and (2.6), we have, for  $\lambda \in \bar{\Gamma}_k(n)$  satisfying (2.5),

$$\sigma_{j;1}(\lambda_1, \dots, \lambda_n) = \sigma_j(\lambda_2, \dots, \lambda_n) \geq 0, \quad \forall j \leq k,$$

which implies

$$(2.7) \quad \lambda^{(1)} := (\lambda_2, \lambda_3, \dots, \lambda_n) \in \bar{\Gamma}_k(n-1).$$

Using Proposition 2.1 for  $\lambda^{(1)} \in \bar{\Gamma}_k(n-1)$ , we have

$$(2.8) \quad \lambda_2 \geq \dots \geq \lambda_{1+k} \geq 0, \quad \sigma_{k-1;2}(\lambda^{(1)}) \geq 0, \quad \sigma_{k-2;2}(\lambda^{(1)}) \geq 0.$$

Using the first equation in (2.6) and the first and third inequalities in (2.8), we have

$$(2.9) \quad 0 = \sigma_{k-1;1}(\lambda) = \sigma_{k-1}(\lambda^{(1)}) = \lambda_2 \sigma_{k-2;2}(\lambda^{(1)}) + \sigma_{k-1;2}(\lambda^{(1)}) \geq \sigma_{k-1;2}(\lambda^{(1)}).$$

Then, by (2.8) and (2.9)

$$(2.10) \quad 0 \geq \sigma_{k-1;2}(\lambda^{(1)}) \geq 0.$$

Accordingly, from (2.6), (2.7) and (2.10), we get  $\lambda^{(1)} \in \bar{\Gamma}_k(n-1)$  and

$$\begin{cases} \sigma_{k-1;2}(\lambda^{(1)}) = 0, \\ \sigma_k(\lambda^{(1)}) = 0, \\ \sigma_{k+1}(\lambda^{(1)}) < 0. \end{cases}$$

This completes the proof of Lemma 2.3 for  $s = 0$ . Then, by an induction on the length of  $\lambda^{(s)}$ , we finish the proof of Lemma 2.3.  $\square$

By Proposition 2.1, if  $\lambda \in \bar{\Gamma}_{m-1}(m)$ ,  $m > 1$  and  $\sigma_{m-1}(\lambda) = 0$ , then

$$\sigma_{m-2,i}(\lambda) \geq 0, \quad i = 1, 2, \dots, m.$$

Under additional condition  $\sigma_m(\lambda) < 0$ , we have

**Lemma 2.4.** *Let  $\lambda \in \bar{\Gamma}_{m-1}(m)$ ,  $m > 2$  and  $\sigma_{m-1}(\lambda) = 0$ . If  $\sigma_m(\lambda) < 0$ , then we have*

$$\sigma_{m-2,i}(\lambda) > 0, \quad i = 1, 2, \dots, m.$$

*Proof.* By Proposition 2.1, we have  $\sigma_{m-2,i}(\lambda) \geq 0$ ; the Maclaurin's inequalities (4.5) yields  $\sigma_m(\lambda) \leq 0$ . Thus, we can equivalently say, if the inequality above does not hold, that is,  $\sigma_{m-2,i}(\lambda) = 0$  for some  $i \in \{1, \dots, m\}$ , then

$$\sigma_m(\lambda) = \lambda_1 \cdots \lambda_m = 0.$$

It is enough to prove that  $\sigma_{m-1}(\lambda) = 0 = \sigma_{m-2;1}(\lambda) = 0$  imply  $\sigma_m(\lambda) = 0$ , since the other case can be deduced by absurd argument of this results.

Substituting  $\sigma_{m-1}(\lambda) = 0 = \sigma_{m-2;1}(\lambda) = 0$  into

$$\sigma_{m-1}(\lambda) = \lambda_1 \sigma_{m-2;1}(\lambda) + \sigma_{m-1;1}(\lambda),$$

we have

$$0 = \sigma_{m-1;1}(\lambda_1, \dots, \lambda_m) = \sigma_{m-1}(\lambda_2, \dots, \lambda_m) = \prod_{i=2}^m \lambda_i.$$

Thus,

$$0 = \lambda_1 \prod_{i=2}^m \lambda_i = \sigma_m(\lambda).$$

$\square$

**Proof of Theorem 2.2.** By these two lemmas above, we prove Theorem 2.2 by an induction on  $k$ . Let  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbb{R}^n$  and  $\lambda$  is in the descending order.

**Step 1: The case  $k = 2$ .** We claim the following results :

*Let  $\lambda \in \bar{\Gamma}_2(n)$  and  $\sigma_2(\lambda) = 0$ , if  $\sigma_3(\lambda) < 0$ , then we have*

$$\sigma_{1,i}(\lambda) > 0, \quad i = 1, 2, \dots, n.$$

*Equivalently, for  $\lambda \in \bar{\Gamma}_2(n)$  with  $\sigma_2(\lambda) = 0$ , if  $\sigma_{1,i}(\lambda) = 0$  for some  $i \in \{1, \dots, n\}$ , then*

$$\sigma_j(\lambda) = 0, \quad j = 3, \dots, n.$$

By assumption above, we have

$$\begin{aligned}\sigma_2(\lambda) &= \lambda_1 \sigma_{1;1}(\lambda) + \sigma_{2;1}(\lambda), \\ \sigma_{1,1}(\lambda) &= \sigma_1(\lambda) - \lambda_1 = \sum_{j=2}^n \lambda_j = \sigma_1(\lambda_2, \dots, \lambda_n), \\ \sigma_{2;1}(\lambda) &= \sigma_2(\lambda_2, \dots, \lambda_n) = \frac{(\sigma_1 - \lambda_1)^2 - \sum_{i=2}^m \lambda_i^2}{2}.\end{aligned}$$

If we assume  $\sigma_{1,1}(\lambda) = 0$ , then  $\sigma_{1,1}(\lambda) = 0$  together with  $\sigma_2(\lambda) = 0$  yields

$$0 = \sigma_2(\lambda) = \lambda_1 \sigma_{1,1}(\lambda) + \sigma_{2,1}(\lambda) = \sigma_{2,1}(\lambda) = \frac{(\sigma_1 - \lambda_1)^2 - \sum_{i=2}^m \lambda_i^2}{2} = \frac{-\sum_{i=2}^m \lambda_i^2}{2},$$

which implies

$$\sum_{i=2}^m \lambda_i^2 = 0,$$

and thus

$$\lambda_i = 0, \quad i = 2, \dots, n.$$

Then

$$\sigma_j(\lambda) = 0, \quad j = 2, \dots, n,$$

which contradicts with  $\sigma_3(\lambda) < 0$ , therefore  $\sigma_{1,1}(\lambda) = 0$  is impossible.

**Step 2: The case  $2 < k \leq n-1$ .** If  $k = n-1$ , it is included in Lemma 2.4. Now we consider the general case  $2 < k < n-1$ .

**The proof of part (I):** We will prove that, for  $2 < k < n-1$ , if  $\lambda \in \bar{\Gamma}_k(n)$ ,  $\sigma_k(\lambda) = 0$  and  $\sigma_{k+1}(\lambda) < 0$ , then

$$\sigma_{k-1;1}(\lambda) > 0.$$

We prove this claim by absurd argument. Recall  $\lambda^{(s)} = (\lambda_{s+1}, \dots, \lambda_n)$  and suppose that  $\lambda = \lambda^{(0)} \in \bar{\Gamma}_k(n)$ ,

$$\begin{cases} \sigma_{k-1;1}(\lambda^{(0)}) = 0, \text{ (the absurd assumption),} \\ \sigma_k(\lambda^{(0)}) = 0, \\ \sigma_{k+1}(\lambda^{(0)}) < 0. \end{cases}$$

By using Lemma 2.3 and the induction assumption up to  $s = n-k-1$ , we have that  $\lambda^{(n-k-1)} \in \bar{\Gamma}_k(k+1)$  and

$$\begin{cases} \sigma_{k-1;n-k}(\lambda^{(n-k-1)}) = \sigma_{k-1;n-k}(\lambda_{n-k}, \dots, \lambda_n) = 0, \\ \sigma_k(\lambda^{(n-k-1)}) = \sigma_k(\lambda_{n-k}, \dots, \lambda_n) = 0, \\ \sigma_{k+1}(\lambda^{(n-k-1)}) = \sigma_{k+1}(\lambda_{n-k}, \dots, \lambda_n) < 0. \end{cases}$$

This contradicts with the conclusion of Lemma 2.4 with  $m = k+1$ . Thus, the assumption  $\sigma_{k-1;1}(\lambda) = 0$  is really absurd, and therefore  $\sigma_{k-1;1}(\lambda) > 0$ .

**The proof of part (II):** We suppose that  $\lambda \in \bar{\Gamma}_k(n)$  and  $\sigma_k(\lambda) = \sigma_{k+1}(\lambda) = 0$ . Then  $\lambda \in \bar{\Gamma}_{k+1}(n)$ , and for any  $\varepsilon > 0$ , from the formula

$$\sigma_{k+1}(\lambda + \varepsilon) = \sum_{j=0}^{k+1} C(j, k, n) \varepsilon^j \sigma_{k+1-j}(\lambda), \quad C(j, k, n) = \frac{\binom{n}{k+1} \binom{k+1}{j}}{\binom{n}{k+1-j}}$$

with the convention  $\sigma_0(\lambda) = 1$ , we have

$$\lambda + \varepsilon = (\lambda_1 + \varepsilon, \lambda_2 + \varepsilon, \dots, \lambda_n + \varepsilon) \in \Gamma_{k+1}(n).$$

We see that either  $\sigma_{k+2}(\lambda) \geq 0$  or  $\sigma_{k+2}(\lambda) < 0$ . Now we prove  $\sigma_{k+2}(\lambda) = 0$ . Firstly we claim that  $\sigma_{k+2}(\lambda) < 0$  is impossible. Otherwise, from the assumption  $\sigma_k(\lambda) = \sigma_{k+1}(\lambda) = 0$ , we have

$$\begin{aligned}\sigma_k(\lambda) &= 0, \\ \sigma_{k+1}(\lambda) &= 0, \\ \sigma_{k+2}(\lambda) &< 0.\end{aligned}$$

Using the results of part (I), we obtain from both  $\sigma_{k+1}(\lambda) = 0$  and  $\sigma_{k+2}(\lambda) < 0$  that

$$(2.11) \quad \sigma_{k;1}(\lambda) > 0.$$

Since  $\lambda \in \bar{\Gamma}_k(n)$ , it is necessary by Proposition 2.1 that  $\lambda_1 \geq 0$  and  $\sigma_{k-1;1}(\lambda) \geq 0$ . So we have

$$0 = \sigma_k(\lambda) = \lambda_1 \sigma_{k-1;1}(\lambda) + \sigma_{k;1}(\lambda) \geq \sigma_{k;1}(\lambda).$$

There is a contradiction of (2.11), thus, the case that  $\sigma_{k+2}(\lambda) < 0$  does not occur and we obtain  $\sigma_{k+2}(\lambda) \geq 0$  in which case  $\lambda \in \bar{\Gamma}_{k+2}(n)$ . Applying the Maclaurin's inequalities to  $\lambda + \varepsilon \in \bar{\Gamma}_{k+2}(n)$ , we obtain

$$\left[ \frac{1}{\binom{n}{k+1}} \sigma_{k+1}(\lambda + \varepsilon) \right]^{\frac{1}{k+1}} \geq \left[ \frac{1}{\binom{n}{k+2}} \sigma_{k+2}(\lambda + \varepsilon) \right]^{\frac{1}{k+2}}.$$

Let  $\varepsilon \rightarrow 0^+$ , then

$$\sigma_{k+1}(\lambda) = 0 \quad \text{implies} \quad \sigma_{k+2}(\lambda) = 0.$$

Repeating the above argument for  $k+1, k+2, \dots, n$ , we obtain

$$\sigma_{k+j}(\lambda) = 0, \quad j = 1, 2, \dots, n-k,$$

that is,  $\lambda \in \bar{\Gamma}_n(n)$  and this completes the proof.  $\square$

Now we are back to the definitions of  $\mathbf{P}_1$  and  $\mathbf{P}_2$ , by Theorem 2.2, can be stated more precisely as, for  $k < n$ ,

$$\mathbf{P}_1 = \{\lambda \in \Gamma_k : \sigma_j(\lambda) \geq 0, \sigma_k(\lambda) = \dots = \sigma_n(\lambda) = 0, 1 \leq j \leq k-1\},$$

$$\mathbf{P}_2 = \{\lambda \in \Gamma_k : \sigma_j(\lambda) > 0, \sigma_k(\lambda) = 0, \sigma_{k+1} < 0, 1 \leq j \leq k-1\}.$$

If  $2 \leq k < n$ , then  $\mathbf{P}_2 \neq \emptyset$ . Here is an example, let

$$\begin{cases} \lambda_i = 1, 1 \leq i \leq k-1 \\ \lambda_k = M, \lambda_{k+1} = -\frac{1}{M} \\ \lambda_i = 0, k+1 < i \leq n \end{cases}$$

with  $M = \frac{k-1 + \sqrt{(k-1)^2 + 4}}{2} > 1$ , then  $M - \frac{1}{M} = k-1$ ,  $\sigma_j(\lambda) > 0 (1 \leq j \leq k-1)$ ,  $\sigma_k(\lambda) = 0$  and  $\sigma_{k+1}(\lambda) = -1$ , which means that  $\mathbf{P}_2 \neq \emptyset$ .

The significance of Theorem 2.2 is the breakthrough of the classical framework of the ellipticity of Hessian equations. It is well known that,  $S_k[u] = f$  is elliptic if  $f > 0$  and degenerate elliptic if  $f \geq 0$ . By the definition of  $\mathbf{P}_2$ , the condition  $f(0) = 0$  must lead to degenerate ellipticity for Monge-Ampère equation. However, it is no longer true for  $k$ -Hessian ( $2 \leq k < n$ ) equation by Theorem 2.2. Next theorem gives a complete  $k$ -Hessian classification of second-order polynomials in the degenerate elliptic case.

**Theorem 2.5.** Suppose that  $\lambda \in \partial\Gamma_k(n)$ ,  $2 \leq k \leq n-1$ .

- (1) For any  $\lambda \in \mathbf{P}_2$ , we have that  $\psi = \frac{1}{2} \sum_{i=1}^n \lambda_i y_i^2$  is a solution of  $k$ -Hessian equation  $S_k[\psi] = 0$ , and the linearized operators of  $\mathcal{L}_\psi = \sum_{i=1}^n \sigma_{k-1,i}(\lambda) \partial_i^2$  is uniformly elliptic.

- (2) For any  $\lambda \in \mathbf{P}_1$  with  $\lambda_1 \geq \dots \geq \lambda_n$ , then  $\psi = \frac{1}{2} \sum_{i=1}^n \lambda_i y_i^2$  is a non-strict convex solution of  $k$ -Hessian equation  $S_k[\psi] = 0$ , and the linearized operators of  $\mathcal{L}_\psi$  is degenerate elliptic with

$$\sigma_{k-1,i} = 0, \quad 1 \leq i \leq k-1.$$

Moreover, if  $\sigma_{k-1}(\lambda) = 0$ , then  $\mathcal{L}_\psi = 0$ , that is,  $\sigma_{k-1,i} = 0$  for  $1 \leq i \leq n$ ; if  $\sigma_{k-1}(\lambda) > 0$ , then

$$(2.12) \quad \begin{cases} \lambda_i > 0, 1 \leq i \leq k-1 \\ \lambda_i = 0, k \leq i \leq n \\ \sigma_{k-1,i} = 0, 1 \leq i \leq k-1 \\ \sigma_{k-1,i} = \prod_{j=1}^{k-1} \lambda_j > 0, k \leq i \leq n \end{cases}$$

*Proof.* (1) is a direct consequence of (I) in Theorem 2.2. Now we prove (2). If  $\lambda \in \mathbf{P}_1$ , we have  $\sigma_{k-1,i} \geq 0$  for  $1 \leq i \leq n$  by (2.3), and  $\sigma_j(\lambda) = 0$  for  $k \leq j \leq n$  by Theorem 2.2 (II). From  $\sigma_n(\lambda) = \prod_{i=1}^n \lambda_i = 0$ ,  $\lambda_j \geq 0$  ( $1 \leq j \leq n$ ) and  $\lambda$  is in decreasing order, it follows that  $\lambda_n = 0$  and

$$\sigma_k(\lambda_1, \dots, \lambda_{n-1}) = \sigma_k(\lambda) - \lambda_n \sigma_{k-1,n}(\lambda) = \sigma_k(\lambda) = 0.$$

Similarly,

$$\sigma_j(\lambda_1, \dots, \lambda_{n-1}) = \sigma_j(\lambda) \geq 0, \sigma_{k+1}(\lambda_1, \dots, \lambda_{n-1}) = \sigma_{k+1}(\lambda) = 0, 1 \leq j \leq k-1.$$

Applying Theorem 2.2 (II) to  $(\lambda_1, \dots, \lambda_{n-1}) \in \Gamma_k(n-1)$ , we obtain

$$\sigma_{n-1}(\lambda_1, \dots, \lambda_{n-1}) = 0$$

and by the same reasoning in the  $n$ -dimensional case,  $\lambda_{n-1} = 0$ . By an induction on the dimension up to  $k+1$ , we see that  $\lambda_i = 0, k+1 \leq i \leq n$ . Since  $\sigma_k(\lambda) = 0$  and

$$\sigma_k(\lambda) = \sigma_k(\lambda_1, \dots, \lambda_k, 0, \dots, 0) = \prod_{i=1}^k \lambda_i,$$

we have  $\lambda_k = 0$  by recalling that  $\lambda$  is in descending order. By the virtue of

$$\sum_{i=1}^n \sigma_{k-1,i}(\lambda) = (n-k+1)\sigma_{k-1}(\lambda), \sigma_{k-1,i}(\lambda) \geq 0, \quad 1 \leq i \leq n,$$

we see that, if  $\sigma_{k-1}(\lambda) = 0$ , then  $\sigma_{k-1,i} = 0$  for  $1 \leq i \leq n$ ; if  $\sigma_{k-1}(\lambda) > 0$ , from

$$0 < \sigma_{k-1}(\lambda) = \sigma_{k-1}(\lambda_1, \dots, \lambda_{k-1}, 0, \dots, 0) = \prod_{i=1}^{k-1} \lambda_i,$$

we have

$$\lambda_i > 0, 1 \leq i \leq k-1.$$

By the definition of  $\sigma_{k-1,i}(\lambda)$  and

$$\lambda = (\lambda_1, \dots, \lambda_{k-1}, 0, \dots, 0),$$

we obtain the last two conclusions of (2.12).  $\square$

If  $\lambda \in \Gamma_k(n)$ , we certainly have  $\lambda \in \Gamma_{k-1}(n)$ . If  $\lambda \in \mathbf{P}_2 \subset \partial\Gamma_k(n)$  with  $\sigma_{k+1}(\lambda) < 0$ , from  $\sum_{i=1}^n \sigma_{k-1,i}(\mu) = (n-k+1)\sigma_{k-1}(\mu)$ ,  $\mu \in \mathbb{R}^n$  and Theorem 2.2 (I), it follows that  $\sigma_{k-1}(\lambda) > 0$  and  $\lambda \in \Gamma_{k-1}(n)$ . In those two cases above,  $0 < \sigma_{k-1;1} \leq \sigma_{k-1;2} \leq \dots \leq \sigma_{k-1;n}$  implies the uniform ellipticity. Conversely, we want to know whether and how the ellipticity is true for  $\lambda \in \Gamma_{k-1}(n)$ . Also notice that, in the monotonicity formula (2.3), it is required that  $\lambda \in \overline{\Gamma_k(n)}$  rather than  $\lambda \in \overline{\Gamma_{k-1}(n)}$  as Lemma 2.6 below.



**Lemma 2.6.** . Suppose that  $\lambda \in \Gamma_{k-1}(n)$ ,  $2 \leq k \leq n-1$ . If  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ , then

$$\sigma_{k-1;1}(\lambda) \geq 0$$

is equivalent to

$$0 \leq \sigma_{k-1;1}(\lambda) \leq \sigma_{k-1;2}(\lambda) \leq \dots \leq \sigma_{k-1;n}(\lambda).$$

*Proof.* We claim that  $\sigma_{k-1;2}(\lambda) \geq \sigma_{k-1;1}(\lambda)$ , that is

$$\sigma_{k-1}(\lambda_1, \lambda_3, \dots, \lambda_n) \geq \sigma_{k-1}(\lambda_2, \lambda_3, \dots, \lambda_n).$$

If the claim is true, by using it again, we obtain

$$\sigma_{k-1;n}(\lambda) \geq \dots \sigma_{k-1;2}(\lambda) \geq \sigma_{k-1;1}(\lambda).$$

Since  $\lambda \in \Gamma_{k-1}(n)$ , by (4.9) we have

$$\sigma_{l;1}(\lambda) > 0, \quad l+1 \leq k-1,$$

which, together with the assumption

$$\sigma_{k-1}(\lambda_2, \lambda_3, \dots, \lambda_n) = \sigma_{k-1,1}(\lambda) \geq 0,$$

yields  $(\lambda_2, \lambda_3, \dots, \lambda_n) \in \Gamma_{k-1}(n-1)$ . Using (4.9) again, we obtain

$$\sigma_{k-2}(\lambda_3, \lambda_4, \dots, \lambda_n) = \sigma_{k-2,2}(\lambda_2, \lambda_3, \dots, \lambda_n) \geq 0.$$

Let  $\varepsilon = \lambda_1 - \lambda_2 \geq 0$ , we have

$$\begin{aligned} \sigma_{k-1;2}(\lambda) &= \sigma_{k-1}(\lambda_1, \lambda_3, \dots, \lambda_n) \\ &= (\lambda_2 + \varepsilon) \sigma_{k-2}(\lambda_3, \dots, \lambda_n) + \sigma_{k-1}(\lambda_3, \dots, \lambda_n) \\ &\geq \lambda_2 \sigma_{k-2}(\lambda_3, \dots, \lambda_n) + \sigma_{k-1}(\lambda_3, \dots, \lambda_n) \\ &= \sigma_{k-1}(\lambda_2, \lambda_3, \dots, \lambda_n) = \sigma_{k-1;1}(\lambda), \end{aligned}$$

thus, the claim is proved.  $\square$

We will give a characterization of ellipticity for the linearized operator of  $S_k[\psi] = c$  with  $c \in \mathbb{R}$ .

**Theorem 2.7.** For any  $c \in \mathbb{R}$ , there exists  $\lambda \in \Gamma_{k-1}(n)$  such that

$$(2.13) \quad 0 < \sigma_{k-1;1}(\lambda) \leq \sigma_{k-1;2}(\lambda) \leq \dots \leq \sigma_{k-1;n}(\lambda),$$

and  $\psi = \frac{1}{2} \sum_{i=1}^n \lambda_i y_i^2$  is a  $(k-1)$ -convex solution of  $k$ -Hessian equation  $S_k[\psi] = c$ , moreover, the linearized operators of  $S_k[u]$  at  $\psi$

$$(2.14) \quad \mathcal{L}_\psi = \sum_{i=1}^n \sigma_{k-1,i}(\lambda) \partial_i^2$$

is uniformly elliptic.

*Proof.* If  $c > 0$ , we take  $\lambda_1 = \lambda_2 = \dots = \lambda_n = [\frac{c}{\binom{n}{k}}]^{\frac{1}{k}} > 0$ , then  $\sigma_k(\lambda) = c$ ; If  $c = 0$ , see Theorem 2.5 (1). It is left to consider the case  $c < 0$ .

Notice that  $\mathbf{P}_2 \neq \emptyset$  when  $k < n$ , so we can choose  $\delta = (\delta_2, \dots, \delta_n) \in \mathbf{P}_2 \subset \partial \Gamma_{k-1}(n-1)$  with  $\delta_2 \geq \delta_3 \geq \dots \geq \delta_n$ , that is,

$$\sigma_{k-1}(\delta_2, \dots, \delta_n) = 0, \sigma_k(\delta_2, \dots, \delta_n) < 0$$

Obviously  $\delta_2 > 0$ . Choosing  $1 > t > 0$  small such that, for  $\delta_1 = \delta_2 + 1$ ,

$$\begin{aligned} \sigma_{k-1}(\delta_2 + t, \dots, \delta_n + t) &> 0, \\ \delta_1 \sigma_{k-1}(\delta_2 + t, \dots, \delta_n + t) + \sigma_k(\delta_2 + t, \dots, \delta_n + t) &< 0. \end{aligned}$$

Then

$$\sigma_k(\delta_1, \delta_2 + t, \dots, \delta_n + t) = \delta_1 \sigma_{k-1}(\delta_2 + t, \dots, \delta_n + t) + \sigma_k(\delta_2 + t, \dots, \delta_n + t) < 0.$$

Since  $\sigma_k(s\lambda) = s^k \sigma_k(\lambda)$  for  $s > 0$ , we can choose suitable  $s > 0$  such that

$$\sigma_k(s\delta_1, s(\delta_2 + t), \dots, s(\delta_n + t)) = c.$$

Let  $\lambda = (s\delta_1, s(\delta_2 + t), \dots, s(\delta_n + t))$ , then

$$\sigma_k(\lambda) = c.$$

The fact  $(\lambda_2, \lambda_3, \dots, \lambda_n) \in \Gamma_{k-1}(n-1)$  and (4.9) lead to

$$\sigma_l(\lambda_2, \lambda_3, \dots, \lambda_n) > 0, \quad 1 \leq l \leq k-1.$$

Therefore, by virtue of  $\lambda_1 = s\delta_1 > 0$ ,

$$\sigma_1(\lambda) = \lambda_1 + \sigma_1(\lambda_2, \lambda_3, \dots, \lambda_n) > \sigma_1(\lambda_2, \lambda_3, \dots, \lambda_n) > 0$$

and

$$\sigma_l(\lambda) = \lambda_1 \sigma_{l-1}(\lambda_2, \lambda_3, \dots, \lambda_n) + \sigma_l(\lambda_2, \lambda_3, \dots, \lambda_n) > 0, \quad 2 \leq l \leq k-1.$$

By the definition of  $\Gamma_{k-1}(n)$ , we have proved that  $\lambda \in \Gamma_{k-1}(n)$ . Noticing

$$\lambda \in \Gamma_{k-1}(n), \quad \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n, \quad \sigma_{k-1}(\lambda_2, \lambda_3, \dots, \lambda_n) > 0$$

and applying Lemma 2.6, we see that

$$0 < \sigma_{k-1;1}(\lambda) \leq \sigma_{k-1;2}(\lambda) \leq \dots \leq \sigma_{k-1;n}(\lambda),$$

which leads to that the operator (2.14) is uniformly elliptic. This completes the proof.  $\square$

**Theorem 2.8.** *For any  $0 < c$ , there exists  $\lambda \in \Gamma_k(n)$  such that*

$$\begin{cases} 0 < \sigma_{k-1;i}(\lambda), & 1 \leq i \leq n \\ \sigma_{k+l-1}(\lambda) > 0, \sigma_{k+l}(\lambda) < 0, & 1 \leq l \leq n-k. \end{cases}$$

*In particular, there exists  $\lambda \in \Gamma_n(n)$  such that*

$$\sigma_k(\lambda) = c.$$

*Therefore, for  $1 \leq l \leq n-k$ ,  $\psi = \frac{1}{2} \sum_{i=1}^n \lambda_i y_i^2$  is a  $(k+l-1)$ -convex solution of  $k$ -Hessian equation  $S_k[\psi] = c$  which is not  $(k+l)$ -convex. Moreover, the linearized operators of  $S_k[\cdot]$  at  $\psi$*

$$\mathcal{L}_\psi = \sum_{i=1}^n \sigma_{k-1,i}(\lambda) \partial_i^2$$

*is uniformly elliptic.*

*Proof.* In the proof of Theorem 2.7, replacing  $\sigma_k$  by  $\sigma_{k+l}$ , we obtain that  $\lambda \in \Gamma_{k+l-1}(n)$  with  $\sigma_{k+l}(\lambda) < 0$  for  $1 \leq l \leq n-k$ . For this  $\lambda$ , choosing  $s > 0$  such that  $\sigma_k(s\lambda) = c$ . The other part of proof is the same as those in Theorem 2.7.  $\square$

3. EXISTENCE OF  $C^\infty$  LOCAL SOLUTIONS

In this section, by the classification of precedent section, we now prove Theorem 1.1 which is stated in the following precise version.

**Theorem 3.1.** *For  $2 \leq k \leq n-1$ , let  $f = f(y, u, p)$  be defined and continuous near a point  $Z_0 = (0, 0, 0) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$  and  $0 < \alpha < 1$ . Assume that  $f$  is  $C^\alpha$  with respect to  $y$  and  $C^{2,1}$  with respect to  $u, p$ . We have the following results*

- (1) *If  $f(Z_0) = 0$ , then (1.1) admits a  $(k-1)$ -convex local solution  $C^{2,\alpha}$  near  $y_0 = 0$ , which is not  $(k+1)$ -convex and of the following form*

$$(3.1) \quad u(y) = \frac{1}{2} \sum_{i=1}^n \tau_i y_i^2 + \varepsilon' \varepsilon^4 w(\varepsilon^{-2} y)$$

*with arbitrarily fixed  $(\tau_1, \dots, \tau_n) \in \mathbf{P}_2$ ,  $\varepsilon'$  is defined in (1.6) and  $\varepsilon > 0$  very small, the function  $w$  satisfies*

$$(3.2) \quad \begin{cases} \|w\|_{C^{2,\alpha}} \leq 1 \\ w(0) = 0, \nabla w(0) = 0. \end{cases}$$

- (2) *If  $f \geq 0$  near  $Z_0$ , then the equation (1.1) admits a  $k$ -convex local solution  $u \in C^{2,\alpha}$  near  $y_0 = 0$  which is not  $(k+1)$ -convex and of the form (3.1).*

*Moreover, the equation (1.1) is uniformly elliptic with respect to the solution (3.1). If  $f \in C^\infty$  near  $Z_0$ , then  $u \in C^\infty$  near  $y_0$ .*

Theorem 3.1 is exactly the part (1) and (2) of Theorem 1.1.

We now proceed to take the change of unknown function  $u \leftrightarrow w$  and the change of variable  $y \leftrightarrow x$ , the aim is to consider the equation (1.1) in the domain  $B_1(0)$  with the new variable  $x$  and then the so-called "local" enter into the new equation itself, see (3.3)-(3.4). The trick is from Lin [11]. Let  $\tau = (\tau_1, \dots, \tau_n) \in \mathbf{P}_2$ , then  $\psi(y) = \frac{1}{2} \sum_{i=1}^n \tau_i y_i^2$  is a polynomial-type solution of

$$S_k[\psi] = 0.$$

We follow Lin [11] to introduce the following function

$$u(y) = \frac{1}{2} \sum_{i=1}^n \tau_i y_i^2 + \varepsilon' \varepsilon^4 w(\varepsilon^{-2} y) = \psi(y) + \varepsilon' \varepsilon^4 w(\varepsilon^{-2} y), \quad \tau \in \mathbf{P}_2, \quad \varepsilon > 0,$$

as a candidate of solution for equation (1.2). Noting  $y = \varepsilon^2 x$ , we have

$$(D_{y_j} u)(x) = \tau_j \varepsilon^2 x_j + \varepsilon' \varepsilon^2 w_j(x), \quad j = 1, \dots, n,$$

and

$$(D_{y_j y_k} u)(x) = \delta_k^j \tau_j + \varepsilon' w_{jk}(x), \quad j, k = 1, \dots, n,$$

where  $\delta_k^j$  is the Kronecker symbol,  $w_j(x) = (D_{y_j} w)(x)$  and  $w_{jk}(x) = (D_{y_j y_k}^2 w)(x)$ . Then (1.1) transfers to

$$\tilde{S}_k(w) = \tilde{f}_\varepsilon(x, w(x), Dw(x)), \quad x \in B_1(0) = \{x \in \mathbb{R}^n; |x| < 1\},$$

where

$$\tilde{S}_k[w] = S_k(\delta_i^j \tau_i + \varepsilon' w_{ij}(x)) = S_k(r(w)),$$

with symmetric matrix  $r(w) = (\delta_i^j \tau_i + \varepsilon' w_{ij}(x))$ , and

$$\tilde{f}_\varepsilon(x, w(x), Dw(x)) = f(\varepsilon^2 x, \varepsilon^4 \psi(x) + \varepsilon' \varepsilon^4 w(x), \tau_1 \varepsilon^2 x_1 + \varepsilon' \varepsilon^2 w_1(x), \dots, \tau_n \varepsilon^2 x_n + \varepsilon' \varepsilon^2 w_n(x)).$$

We now explain the smooth condition on  $f = f(x, u, p)$  and its norms, that is,  $f$  is Hölder continuous with respect to  $x$ , denoted by  $f \in C_x^\alpha$ , and  $C^{2,1}$  with respect to  $u, p$ , denoted by  $C_{u,p}^{2,1}$ . We will consider  $f = f(x, u, p)$  defined on

$$\mathcal{B} = \{(x, u, p) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n : x \in B_1(0), |u| \leq A, |p| \leq A\}$$

for some fixed  $A > 0$ . We say  $f \in C_x^\alpha$  if

$$\|f\|_{C_x^\alpha} = \|f\|_{L^\infty(\mathcal{B})} + \sup_{(x,u,p) \in \mathcal{B}, (z,u,p) \in \mathcal{B}, x \neq z} \frac{|f(x, u, p) - f(z, u, p)|}{|x - z|^\alpha} < \infty.$$

When  $f = f(x)$ , then  $f \in C_x^\alpha$  is the usual  $f \in C^\alpha(B_1(0))$ . When defining  $f = f(z, u, p) \in C_{u,p}^{2,1}$ , we regard  $z$  as a parameter as follow:

$$\|f\|_{C_{u,p}^{1,1}} = \sup_{\mathcal{B}} \{|D_{u,p}^\beta f(z, u, p)| : 0 \leq |\beta| \leq 2\} < \infty$$

and

$$\|f\|_{C_{u,p}^{2,\alpha}} = \|f\|_{C_{u,p}^{1,1}} + \sup_{(z,u,p) \in \mathcal{B}, (z,u',p') \in \mathcal{B}, (u,p) \neq (u',p')} \left\{ \frac{|D_{u,p}^\beta f(z, u, p) - D_{u,p}^\beta f(z, u', p')|}{|(u, p) - (u', p')|^\alpha}, |\beta| = 2 \right\} < \infty,$$

for  $0 < \alpha < 1$  and a similar definition for  $\|f\|_{C_{u,p}^{2,1}}$ . Here and later on, without confusion, we will denote  $\|f\|_{C_x^\alpha}$ ,  $\|f\|_{C_{u,p}^{1,1}}$  and  $\|f\|_{C_{u,p}^{2,1}}$  as  $\|f\|_{C^\alpha}$ ,  $\|f\|_{C^{1,1}}$  and  $\|f\|_{C^{2,1}}$  respectively.

Similar to [11] we consider the nonlinear operators

$$(3.3) \quad G(w) = \frac{1}{\varepsilon'} [S_k(r(w)) - \tilde{f}_\varepsilon(x, w, Dw)], \quad \text{in } B_1(0).$$

The linearized operator of  $G$  at  $w$  is

$$(3.4) \quad L_G(w) = \sum_{i,j=1}^n \frac{\partial S_k(r(w))}{\partial r_{ij}} \partial_{ij}^2 + \sum_{i=1}^n a_i \partial_i + a,$$

where

$$a_i = -\frac{1}{\varepsilon'} \frac{\partial \tilde{f}_\varepsilon(x, z, p_i)}{\partial p_i}(x, w, Dw) = -\varepsilon^2 \frac{\partial f}{\partial p_i}$$

$$a = -\frac{1}{\varepsilon'} \frac{\partial \tilde{f}_\varepsilon(x, z, p_i)}{\partial z}(x, w, Dw) = -\varepsilon^4 \frac{\partial f}{\partial z}.$$

Hereafter, we denote  $S_k^{ij}(r(w)) = \frac{\partial S_k(r(w))}{\partial r_{ij}}$ .

**Lemma 3.2.** *Assume that  $\tau \in \mathbf{P}_2$  and  $\|w\|_{C^2(B_1(0))} \leq 1$ , then the operator  $L_G(w)$  is a uniformly elliptic operator if  $\varepsilon > 0$  is small enough.*

*Proof.* In order to prove the uniform ellipticity of  $L_G(w)$

$$\sum_{i,j=1}^n S_k^{ij}(r(w)) \xi_i \xi_j \geq c |\xi|^2, \quad \forall (x, \xi) \in B_1(0) \times \mathbb{R}^n,$$

it suffices to prove

$$(3.5) \quad \begin{cases} S_k^{ii}(r(w)) = \sigma_{k-1,i}(\tau_1, \tau_2, \dots, \tau_n) + O(\varepsilon'), & 1 \leq i \leq n \\ S_k^{ij}(r(w)) = O(\varepsilon'), & i \neq j, \end{cases}$$

because if it does hold, we see by (2.13) that

$$S_k^{ii}(r(w)) - \sum_{j=1, j \neq i}^n |S_k^{ij}(r(w))| > \frac{1}{2} \sigma_{k-1,i}(\tau_1, \dots, \tau_n) > 0, \quad 1 \leq i \leq n$$

if  $\varepsilon > 0$  is small enough, then the matrix  $(S_k^{ij}(r(w)))$  is strictly diagonally dominant and  $L_G(w)$  is a uniformly elliptic operator.

Indeed, Since  $S_k(r)$  is the sum all principal minors of order  $k$  of the Hessian  $\det(r)$ , then

$$S_k^l(r(w)) = S_{k-1}(r(w; l, l))$$

where  $r(w; l, l)$  is a  $(n-1) \times (n-1)$  matrix determined from  $r$  by deleting the  $l$ -th row and  $l$ -th column. But  $r(w) = (\delta_i^j \tau_i + \varepsilon w_{ij}(x))$ , then

$$S_{k-1}(r(w; l, l)) = \sigma_{k-1;l}(\tau_1, \tau_2, \dots, \tau_n) + O(\varepsilon')$$

and

$$S_k^{ij}(r(w)) = O(\varepsilon'), \quad i \neq j.$$

Proof is done.  $\square$

We follows now the idea of Hong and Zuily [7] to prove the existence and a priori estimates of solution for linearized operator. In fact, if following the proof of [7] step by step, we can also obtain the existence of the local solution if  $f$  is smooth enough, the reason is that  $L_G(w)$  being uniformly elliptic can regarded as a special case of  $L_G(w)$  being degenerately elliptic in [7]. But if  $f \in C_x^\alpha$ , which is the least requirement in classical Schauder estimates, their proof [7] does not work anymore because the degeneracy results in the loss of regularity. In our case, although  $L_G(w)$  is uniformly elliptic, the existence and the priori Schauder estimates of classical solutions can not be directly obtained. The difficulty lies in that we do not know whether the coefficient  $a$  of the term  $au$  in (3.4) is non-positive. After proving the existence of the linearized equation (Lemma 3.3), we can employ Nash-Moser procedure to prove the existence of local solution for (1.1) in Hölder space. We shall use the following schema :

$$(3.6) \quad \begin{cases} w_0 = 0, & w_m = w_{m-1} + \rho_{m-1}, \quad m \geq 1, \\ L_G(w_m)\rho_m = g_m, \text{ in } B_1(0), \\ \rho_m = 0 \quad \text{on } \partial B_1(0), \\ g_m = -G(w_m), \end{cases}$$

where

$$g_0(x) = \frac{1}{\varepsilon'} \left( \sigma_k(\tau) - f(\varepsilon^2 x, \varepsilon^4 \psi(x), \varepsilon^2(\tau_1 x_1, \tau_2 x_2, \dots, \tau_n x_n)) \right).$$

It is pointed out on page 107, [3] that, if the operator  $L_G$  does not satisfy the condition  $a \leq 0$ , as is well known from simple examples, the Dirichlet problem for  $L_G(w)\rho = g$  no longer has a solution in general. Notice  $a$  in (3.4) has the factor  $\varepsilon^4$ , we will take advantage of the smallness of  $a$  to obtain the uniqueness and existence of solution for Dirichlet problem (3.7). We will assume  $\|w_m\|_{C^{2,\alpha}} \leq A$  rather than  $\|w_m\|_{C^{2,\alpha}} \leq 1$  as in [7], the advantage is to see how the procedure depends on  $A$ . Actually  $A$  can be taken as 1. We have uniformly Schauder estimates of its solution as follows.

**Lemma 3.3.** *Assume that  $\|w\|_{C^{2,\alpha}(B_1(0))} \leq A$ . Then there exists a unique solution  $\rho \in C^{2,\alpha}(\overline{B_1(0)})$  to the following Dirichlet problem*

$$(3.7) \quad \begin{cases} L_G(w)\rho = g, & \text{in } B_1(0), \\ \rho = 0 & \text{on } \partial B_1(0) \end{cases}$$

for all  $g \in C^\alpha(\overline{B_1(0)})$ . Moreover,

$$(3.8) \quad \|\rho\|_{C^{2,\alpha}(\overline{B_1(0)})} \leq C \|g\|_{C^\alpha(\overline{B_1(0)})}, \quad \forall g \in C^\alpha(\overline{B_1(0)}),$$

where the constant  $C$  depends on  $A, \tau$  and  $\|f\|_{C^{2,1}}$ . Moreover,  $C$  is independent of  $0 < \varepsilon \leq \varepsilon_0$  for some  $\varepsilon_0 > 0$ .

By the virtue of (3.4), we write (3.7) as

$$(3.9) \quad \begin{cases} L_G(w)\rho = \sum_{i,j=1}^n \frac{\partial S_k(r(w))}{\partial r_{ij}} \partial_i \partial_j \rho + \sum_{i=1}^n a_i \partial_i \rho + a\rho = g, & \text{in } B_1(0), \\ \rho = 0 & \text{on } \partial B_1(0) \end{cases}$$

where

$$a_i = -\varepsilon^2 \frac{\partial f}{\partial p_i}, \quad a = -\varepsilon^4 \frac{\partial f}{\partial z}.$$

Noticing that for the functions, such as  $\frac{\partial S_k(r(w))}{\partial r_{ij}}$ ,  $a_i = a_i(x, w(x), Dw(x))$ ,  $a = a(x, w(x), Dw(x))$  and  $g_m = -G(w_m) = g_m(x, w_m(x), Dw_m(x), D^2 w_m(x))$  by (3.6), we regard them as the functions with variable  $x$ . In a word, we regard that all of the coefficients and non-homogeneous term in (3.9) are functions of variable  $x$ . For example,

$$\begin{aligned} & \tilde{f}_\varepsilon(x, w(x), Dw(x)) \\ &= f(\varepsilon^2 x, \varepsilon^4 \psi(x) + \varepsilon' \varepsilon^4 w(x), \tau_1 \varepsilon^2 x_1 + \varepsilon' \varepsilon^2 w_1(x), \dots, \tau_n \varepsilon^2 x_n + \varepsilon' \varepsilon^2 w_n(x)). \end{aligned}$$

*Proof of Lemma 3.3.* Let

$$\mu(\tau) = \inf \left\{ \sum_{i,j=1}^n S_k^{ij}(r(w)) \xi_i \xi_j, \quad \forall x \in B_1(0), |\xi| = 1, \|w\|_{C^{2,\alpha}(\overline{B_1(0)})} \leq A \right\}.$$

By Lemma 3.2,  $\mu(\tau) > 0$ . Applying Theorem 3.7 in [3] to the solution  $u \in C^0(\overline{B_1(0)}) \cap C^2(B_1(0))$  of

$$\begin{cases} L_G(w)u = \sum_{i,j=1}^n \frac{\partial S_k(r(w))}{\partial r_{ij}} \partial_i \partial_j u + \sum_{i=1}^n a_i \partial_i u = g, & \text{in } B_1(0), \\ u = 0, & \text{on } \partial B_1(0), \end{cases}$$

we have

$$(3.10) \quad \sup |u| \leq \frac{C}{\mu(\tau)} \|g\|_{C^0(\overline{B_1(0)})},$$

where  $C = e^{2(\beta+1)} - 1$  and  $\beta = \sup \left\{ \frac{|a_i|}{\mu(\tau)} : i = 1, 2, \dots, n \right\}$ .

Let  $C_1 = 1 - C \sup \frac{|a_i|}{\mu(\tau)}$  with  $C$  being the constant in (3.10). If we choose  $\varepsilon_0 > 0$  small (since  $a = O(\varepsilon^4)$  is small), then  $C_1 > \frac{1}{2}$  is independent of  $0 < \varepsilon < \varepsilon_0$ . By applying Corollary 3.8 in [3] to the solution  $\rho$  to Dirichlet problem (3.9), we have

$$(3.11) \quad \sup |\rho| \leq \frac{1}{C_1} \left[ \sup_{\partial B_1(0)} |\rho| + \frac{C}{\mu(\tau)} \|g\|_{C^0(\overline{B_1(0)})} \right] = \frac{C}{C_1 \mu(\tau)} \|g\|_{C^0(\overline{B_1(0)})}.$$

From (3.11) we see that the homogeneous problem

$$\begin{cases} L_G(w)\rho = \sum_{i,j=1}^n \frac{\partial S_k(r(w))}{\partial r_{ij}} \partial_i \partial_j \rho + \sum_{i=1}^n a_i \partial_i \rho + a\rho = 0, & \text{in } B_1(0), \\ \rho = 0 & \text{on } \partial B_1(0) \end{cases}$$

only possesses the trivial solution. Then we can apply a Fredholm alternative, Theorem 6.15 in [3], to the inhomogeneous problem (3.9) for which we can assert that it has a unique  $C^{2,\alpha}(\overline{B_1(0)})$  solution for all  $g \in C^\alpha(\overline{B_1(0)})$ .

With the existence and uniqueness at hand, we can apply Theorem 6.19 [3] to obtain higher regularity up to boundary for solution to (3.9). Besides this, we have the Schauder estimates (see Problem 6.2, [3])

$$(3.12) \quad \|\rho\|_{C^{2,\alpha}} \leq C(A, \tau, \|f\|_{C^{1,1}}) \left[ \|\rho_k\|_{C^0(\overline{B_1(0)})} + \|g_k\|_{C^\alpha(\overline{B_1(0)})} \right],$$

where  $C$  depends on  $C^\alpha$ -norm of all of the coefficients; the uniform ellipticity; boundary value and boundary itself. Now we explain the dependence of  $C(A, \tau, \|f\|_{C^{1,1}})$ . Firstly, since

the first two derivatives of  $w$  have come into the principal coefficients  $\frac{\partial S_2(r(w))}{\partial r_{ij}}$ , then their  $C^\alpha$ -norms must be involved in  $\|w\|_{C^{2,\alpha}}$ . That is,  $\|w\|_{C^{2,\alpha}} \leq A$  arise into  $C$ . Similarly, by the virtue of the coefficients  $a_i$  and  $a$ , we have that  $\|f\|_{C^{1,1}}$  and  $\|w\|_{C^{2,\alpha}} \leq A$  must arise into  $C$ . Secondly, it depends on the uniform ellipticity, that is,

$$\inf \left\{ \sum_{i,j=1}^n S_k^{ij}(r(w)\xi_i\xi_j), \quad \forall x \in B_1(0), |\xi| = 1, \|w\|_{C^{2,\alpha}(B_1(0))} \leq A \right\}$$

and

$$\sup \left\{ \sum_{i,j=1}^n S_k^{ij}(r(w)\xi_i\xi_j), \quad \forall x \in B_1(0), |\xi| = 1, \|w\|_{C^{2,\alpha}(B_1(0))} \leq A \right\},$$

so  $(\tau = (\tau_1, \tau_2, \dots, \tau_n))$  and  $A$  arise into  $C$ .

Thirdly, Since its boundary value is zero and boundary  $\partial B_1(0)$  is  $C^\infty$ , the these two ingredients do not occur into  $C$ . Substituting (3.11) into (3.12), we obtain (3.8).  $\square$

It follows from the standard elliptic theory (see Theorem 6.17 in [3] and Remark 2 in [1]) and an iteration argument that we obtain.

**Corollary 3.4.** *Assume that  $u \in C^{2,\alpha}(\Omega)$  is a solution of (1.1), and the linearized operators with respect to  $u$ ,*

$$\mathcal{L}_u = \sum_{i,j=1}^n \frac{\partial S_k(u_{ij})}{\partial r_{ij}} \partial_{ij}^2 - \sum_{i=1}^n \frac{\partial f}{\partial p_i}(y, u(y), Du(y)) \partial_i - \frac{\partial f}{\partial z}(y, u(y), Du(y)),$$

*is uniformly elliptic. If  $f \in C^\infty$ , then  $u \in C^\infty(\Omega)$ .*

Using Lemma 3.3 above, we can use the procedure (3.6) to construct the sequence  $\{w_m\}_{m \in \mathbb{N}}$ . Now we study the convergence of  $\{w_m\}_{m \in \mathbb{N}}$  and  $\{g_m\}_{m \in \mathbb{N}}$ .

**Proposition 3.5.** *Let  $\{w_m\}_{m \in \mathbb{N}}$  and  $\{g_m\}_{m \in \mathbb{N}}$  be the sequence in (3.6). Suppose that  $\|w_j\|_{C^{2,\alpha}} \leq A$  for  $j = 1, 2, \dots, l$ . Then we have*

$$(3.13) \quad \|g_{l+1}\|_{C^\alpha} \leq C \|g_l\|_{C^\alpha}^2,$$

*where  $C$  is some positive constant depends only on  $\tau$ ,  $A$  and  $\|f\|_{C^{2,1}}$ . In particular,  $C$  is independent of  $l$ .*

*Proof.* By applying Taylor expansion with integral-typed remainder to (3.3), we have

$$\begin{aligned} -g_{l+1} &= G(w_l + \rho_l) = G(w_l) + L_G(w_l)\rho_l + Q(w_l, \rho_l) \\ &= -g_l + L_G(w_l)\rho_l + Q(w_l, \rho_l) = Q(w_l, \rho_l), \end{aligned}$$

where  $Q$  is the quadratic error of  $G$  which consists of  $S_k$  and  $f$ ,

$$\begin{aligned} Q(w_l, \rho_l) &= \sum_{i,j,st} \frac{1}{\varepsilon} \int (1-\mu) \frac{\partial^2 S_k(w_l + \mu\rho_l)}{\partial w_{ij} \partial w_{st}} d\mu (\rho_l)_{ij} (\rho_l)_{st} \\ &\quad - \sum_{i,j} \frac{1}{\varepsilon} \int (1-\mu) \frac{\partial^2 \tilde{f}_\varepsilon(w_l + \mu\rho_l)}{\partial w_i \partial w_j} d\mu (\rho_l)_i (\rho_l)_j \\ &\quad - \frac{1}{\varepsilon} \sum_i \int (1-\mu) \frac{\partial^2 \tilde{f}_\varepsilon(w_l + \mu\rho_l)}{\partial w \partial w_i} d\mu (\rho_l)_i (\rho_l) \\ &\quad - \frac{1}{\varepsilon} \int (1-\mu) \frac{\partial^2 \tilde{f}_\varepsilon(w_l + \mu\rho_l)}{\partial w^2} d\mu \cdot \rho_l^2 \\ &= I_1 + I_2 + I_3 + I_4 \end{aligned} \tag{3.14}$$

Since  $S_k(r(w))$  is a  $k$ -order homogeneous polynomial with variable  $r_{ij}(r(w))$  and  $\tilde{f}_\varepsilon(x, w, Dw)$  is independent of  $r_{ij}$ , we see that

$$\begin{aligned} \left| \frac{\partial^2 S_k(w_l + \mu \rho_l)}{\partial w_{ij} \partial w_{st}} \right| &= \varepsilon'^2 \left| \frac{\partial^2 S_k}{\partial w_{ij} \partial w_{st}} (\delta_i^j \tau_i + \varepsilon' (w_l + \mu \rho_l)_{ij}) \right| \\ &= \varepsilon'^2 \sum_{j=2}^k C(j, \tau) [\partial^2 (w_l + \mu \rho_l)]^{j-2} \\ \left| \frac{\partial^2 \tilde{f}_\varepsilon(w_l + \mu \rho_l)}{\partial w_i \partial w_j} \right| &= \left| \frac{\partial^2 [f(\varepsilon x, \varepsilon^4 \psi + \varepsilon' \varepsilon^4 (w_l + \mu \rho_l), \varepsilon^2 D\psi + \varepsilon' \varepsilon^2 D(w_l + \mu \rho_l))]}{\partial w_i \partial w_j} \right| \\ &\leq \varepsilon'^2 \varepsilon^4 \cdot \|f\|_{C^{1,1}}, \\ \left| \frac{\partial^2 \tilde{f}_\varepsilon(w_l + \mu \rho_l)}{\partial w \partial w_i} \right| &= \left| \frac{\partial^2 [f(\varepsilon x, \varepsilon^4 \psi + \varepsilon' \varepsilon^4 (w_l + \mu \rho_l), \varepsilon^2 D\psi + \varepsilon' \varepsilon^2 D(w_l + \mu \rho_l))]}{\partial w \partial w_i} \right| \\ &\leq \varepsilon'^2 \varepsilon^6 \|f\|_{C^{1,1}}, \\ \left| \frac{\partial^2 \tilde{f}_\varepsilon(w_l + \mu \rho_l)}{\partial w^2} \right| &= \left| \frac{\partial^2 [f(\varepsilon x, \varepsilon^4 \psi + \varepsilon' \varepsilon^4 (w_l + \mu \rho_l), \varepsilon^2 D\psi + \varepsilon' \varepsilon^2 D(w_l + \mu \rho_l))]}{\partial w^2} \right| \\ &= \varepsilon'^2 \varepsilon^8 \|f\|_{C^{1,1}}. \end{aligned}$$

Thus,  $I_i (1 \leq i \leq 4)$  in  $\mathcal{Q}$  are under control by  $O(\varepsilon')$ ,  $O(\varepsilon' \varepsilon^4)$ ,  $O(\varepsilon' \varepsilon^6)$  and  $O(\varepsilon' \varepsilon^8)$  respectively. Therefore

$$\|I_1\|_{C^\alpha} \leq C \sum_{j=1}^{k-1} \|\rho_l\|_{C^2}^j \|\rho_l\|_{C^{2,\alpha}}$$

and

$$\begin{aligned} \|I_2\|_{C^\alpha} &\leq C \|f\|_{C^\alpha} (\|w_l\|_{C^{1,\alpha}} + \|\rho_l\|_{C^{1,\alpha}}) \|\rho_l\|_{C^1}^2 + C \|f\|_{C^{1,1}} \|\rho_l\|_{C^{2,\alpha}} \|\rho_l\|_{C^1} \\ &\leq C \|\rho_l\|_{C^{2,\alpha}} \|\rho_l\|_{C^1}^2 + C \|\rho_l\|_{C^1}^2 + C \|\rho_l\|_{C^{2,\alpha}} \|\rho_l\|_{C^1} \end{aligned}$$

holds, where  $C$  depends on  $A$  and  $\|f\|_{C^{2,\alpha}}$ . And  $\|I_3\|_{C^\alpha}$  and  $\|I_4\|_{C^\alpha}$  can be estimated similarly. Accordingly,

$$\begin{aligned} \|g_{l+1}\|_{C^\alpha} &= \|\mathcal{Q}(w_l, \rho_l)\|_{C^\alpha} \leq \sum_{i=1}^4 \|I_i\|_{C^\alpha} \\ &\leq C \sum_{j=1}^{k-1} \|\rho_l\|_{C^2}^j \|\rho_l\|_{C^{2,\alpha}} + C \|\rho_l\|_{C^{2,\alpha}} \|\rho_l\|_{C^1}^2 + \|\rho_l\|_{C^1}^2 + C \|\rho_l\|_{C^{2,\alpha}} \|\rho_l\|_{C^1}, \end{aligned}$$

where  $C$  is independent of  $l$  but dependent of  $A$  and  $\|f\|_{C^{2,1}}$ . Thus, by the interpolation inequalities, we have

$$\|g_{l+1}\|_{C^\alpha} \leq C \sum_{j=1}^{k-1} \|\rho_l\|_{C^{2,\alpha}}^{j+1} + C \|\rho_l\|_{C^{2,\alpha}}^3,$$

where  $C$  is independent of  $l$ . By the Schauder estimates of Lemma 3.3, we have

$$\|\rho_l\|_{C^{2,\alpha}} \leq C \|g_l\|_{C^\alpha}.$$

Recall that  $\|g_l\|_{C^\alpha} = \|G(w_m)\|_{C^\alpha} \leq C(A)$  holds provided  $\|w_j\|_{C^{2,\alpha}} \leq A$  for  $j = 1, 2, \dots, l$ . Combining the two estimates above, we obtain (3.13).

For the special case  $f = f(y) = f(\varepsilon^2 x)$ , then  $I_2 = I_3 = I_4 = 0$  in (3.14). So the condition  $f = f(y) \in C^\alpha$  is enough for the estimate (3.16)-(3.17) below. Proof is done.  $\square$



Since  $C$  is independent of  $l$ , more exactly,  $A$ ,  $\tau$  and  $\|f\|_{C^{2,1}}$  are independent of  $l$ . So hereafter, we can assume  $A = 1$ .

**Proof of Theorem 3.1.** Set

$$(3.15) \quad d_{l+1} = C \|g_{l+1}\|_{C^{2,\alpha}}, \quad l = 0, 1, 2, \dots$$

By (3.13) and setting  $C \geq 1$  we have

$$d_{l+1} \leq d_l^2.$$

Take  $\tau \in \mathbf{P}_2$  such that  $\sigma_k(\tau) = f(0, 0, 0)$ , we have

$$\begin{aligned} g_0(x) &= -G(0) = \frac{1}{\varepsilon'} [S_k(r(0)) - \tilde{f}(x, 0, 0)] \\ &= \frac{1}{\varepsilon'} [\sigma_k(\tau) - f(\varepsilon^2 x, \varepsilon^4 \psi(x), \varepsilon^2(\tau_1 x_1, \dots, \tau_n x_n))] \\ &= \frac{1}{\varepsilon'} [(\sigma_k(\tau) - f(0, 0, 0)) + \frac{1}{\varepsilon} [f(0, 0, 0) - f(\varepsilon^2 x, 0, 0)] \\ &\quad + \frac{1}{\varepsilon'} [f(\varepsilon^2 x, 0, 0) - f(\varepsilon^2 x, \varepsilon^4 \psi(x), \varepsilon^2(\tau_1 x_1, \dots, \tau_n x_n))] \\ &= \frac{1}{\varepsilon'} [f(0, 0, 0) - f(\varepsilon^2 x, 0, 0)] \\ &\quad - \frac{\varepsilon^4}{\varepsilon'} \int_0^1 \psi(x) (\partial_z f)(\varepsilon^2 x, t \varepsilon^4 \psi(x), t \varepsilon^2(\tau_1 x_1, \dots, \tau_n x_n)) dt \\ &\quad - \frac{\varepsilon^2}{\varepsilon'} \int_0^1 (\tau_1 x_1, \dots, \tau_n x_n) \cdot (\partial_p f)(\varepsilon^2 x, t \varepsilon^4 \psi(x), t \varepsilon^2(\tau_1 x_1, \dots, \tau_n x_n)) dt, \end{aligned}$$

where  $\sigma_k(\tau) - f(0, 0, 0) = 0$  is used. Noticing

$$\frac{1}{\varepsilon'} \|f(0, 0, 0) - f(\varepsilon^2 x, 0, 0)\|_{C^\alpha(B_1(0))} \leq C \frac{\varepsilon^{2\alpha}}{\varepsilon'} \|f(\cdot, 0, 0)\|_{C^\alpha(B_{\varepsilon^2}(0))},$$

we obtain

$$(3.16) \quad \|g_0\|_{C^\alpha(B_1(0))} \leq C_1 \frac{\varepsilon^{2\alpha}}{\varepsilon'} \|f\|_{C^{1,1}}.$$

Using the definition of  $\varepsilon'$  in (1.6), we can choose  $0 < \varepsilon \leq \varepsilon_0$  so small that

$$(3.17) \quad C \|g_0\|_{C^\alpha(B_1(0))} \leq \frac{1}{4}, \quad 0 < \varepsilon \leq \varepsilon_0.$$

Notice  $\varepsilon_0$  is independent of  $l$ . Since  $d_0 = C \|g_0\|_{C^\alpha}$ , we have  $d_1 \leq d_0^2$ . Then, by an induction,

$$d_{l+1} \leq 2^{2^{l+1}} d_0^{2^{l+1}} \leq (2C)^{2^{l+1}} \|g_0\|_{C^\alpha}^{2^{l+1}}.$$

Thus, by (3.15) and (3.17)

$$(3.18) \quad \|g_{l+1}\|_{C^\alpha} \leq (2C)^{2^{l+1}-1} \|g_0\|_{C^\alpha}^{2^{l+1}} \leq \left(\frac{1}{2}\right)^{2^l} \rightarrow 0.$$

Firstly, we claim that there exists a constant  $\varepsilon_0 > 0$ , depending on  $\tau$  and  $\|f\|_{C^{2,1}}$  such that, uniformly for  $\varepsilon \in (0, \varepsilon_0]$ ,

$$\|w_l\|_{C^{2,\alpha}(B_1(0))} \leq 1, \quad \forall l \geq 1.$$

Indeed, set  $w_0 = 0$ , we have by (3.13)

$$\begin{aligned} \|w_{l+1}\|_{C^{2,\alpha}(B_1(0))} &= \left\| \sum_{i=0}^l \rho_i \right\|_{C^{2,\alpha}(B_1(0))} \leq \sum_{i=0}^l \|\rho_i\|_{C^{2,\alpha}(B_1(0))} \\ &\leq \sum_{i=0}^l C \|g_i\|_{C^\alpha(B_1(0))} \leq \sum_{i=0}^l \left( C \|g_0\|_{C^\alpha(B_1(0))} \right)^{2^i} \end{aligned}$$

where  $C$  is defined in Lemma 3.5. Thus, for any  $l$ ,

$$(3.19) \quad \|w_{l+1}\|_{C^{2,\alpha}(B_1(0))} \leq \sum_{i=0}^{\infty} \left( C \|g_0\|_{C^\alpha(B_1(0))} \right)^{2^i} \leq \sum_{i=0}^{\infty} 2^{-2^i} \leq 1.$$

Then, by Azelà-Ascoli Theorem, there is a subsequence of  $w_l$ , still denoted by  $w_l$ , such that

$$w_l \rightarrow w \quad \text{in } C^2(B_1(0)),$$

and  $w \in C^{2,\alpha}(B_1(0))$ . From (3.18) and  $g_m = -G(w_m)$ , we have

$$G(w) = \frac{1}{\varepsilon'} [S_k(r(w)) - \tilde{f}(x, w, Dw)] = 0, \quad \text{on } B_1(0).$$

That means to say the function

$$u(y) = \frac{1}{2} \sum_{i=1}^n \tau_i y_i^2 + \varepsilon' \varepsilon^4 w(\varepsilon^{-2} y) \in C^{2,\alpha}(B_{\varepsilon^2}(0)),$$

is a solution of

$$S_k[u] = f(y, u, Du), \quad \text{on } B_{\varepsilon^2}(0).$$

Now if  $f(0, 0, 0) = 0$ , we take  $\tau \in \mathbf{P}_2$ , then  $\sigma_{k-1}(\tau) > 0, \sigma_k(\tau) = 0, \sigma_{k+1}(\tau) < 0$ . Noticing that symmetric matrix  $r(w) = (\delta_i^j \tau_i + \varepsilon' w_{ij}(x))$ , we have

$$S_j[u] = \sigma_j(\lambda) = \sigma_j(\tau) + O(\varepsilon'), \quad j = 1, 2, \dots, k+1.$$

it follows that  $S_j[u] > 0 (1 \leq j \leq k-1), S_{k+1}[u] < 0$  on  $B_{\varepsilon^2}(0)$  for small  $\varepsilon > 0$ . That is,  $u$  is  $(k-1)$ -convex but not  $(k+1)$ -convex. Moreover if  $S_k[u] = f \geq 0$  near  $Z_0$  and  $f(Z_0) = 0$ , we see that  $u$  is  $k$ -convex by definition, but not  $(k+1)$ -convex.

If  $S_k[u] = f > 0$  near  $Z_0$ , we use Theorem 2.8 to take  $\tau \in \mathbb{R}^n$  given in  $\Gamma_{k-l+1}(n) \setminus \bar{\Gamma}_{k+l}(n)$  for  $1 \leq l \leq n-k$ , then we can get the  $(k+l-1)$ -convex but not  $(k+l)$ -convex local solutions. From (3.19) and condition  $Z_0 = (0, 0, 0)$ , we obtain (3.2).

The  $C^\infty$  regularity of solution is given by Corollary 3.4. Thus, we have proved Theorem 3.1.  $\square$

We also have the following elliptic results for  $f(Z_0) < 0$  which is (3) in the Theorem 1.1.

**Theorem 3.6.** *For  $2 \leq k \leq n-1$ , let  $f = f(y, u, p)$  be defined and continuous near a point  $Z_0 = (0, 0, 0) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$  and  $0 < \alpha < 1$ . Assume that  $f$  is  $C^\alpha$  with respect to  $y$  and  $C^{2,1}$  with respect to  $u, p$ . If  $f(Z_0) < 0$ , then (1.1) admits a  $(k-1)$ -convex local solution  $C^{2,\alpha}$  near  $y_0 = 0$ , which is not  $k$ -convex and of the following form*

$$u(y) = \frac{1}{2} \sum_{i=1}^n \tau_i y_i^2 + \varepsilon' \varepsilon^5 w(\varepsilon^{-2} y)$$

with  $\sigma_k(\tau_1, \dots, \tau_n) = f(0, 0, 0)$ ,  $\varepsilon > 0$  very small,  $\varepsilon'$  is defined in (1.6) and  $w$  satisfies (3.2). Moreover, the equation (1.1) is uniformly elliptic with respect to the solution above. If  $f \in C^\infty$  near  $Z_0$ , then  $u \in C^\infty$  near  $y_0$ .

*Proof.* For  $f(0, 0, 0) < 0$ , take  $\tau \in \mathbb{R}^n$  as in Theorem 2.7 with  $c = f(0, 0, 0) < 0$  such that

$$\sigma_{k-1}(\tau) > 0, \quad \sigma_k(\tau) = f(0, 0, 0) < 0,$$

and

$$\sigma_{k-1;n}(\tau) \geq \sigma_{k-1;n-1}(\tau) \geq \dots \sigma_{k-1,1}(\tau) > 0.$$

Now the proof is exactly the same as that of Theorem 3.1.  $\square$

#### 4. APPENDIX

The following results are essential in the proof of our main theorem, maybe it is classical, but we can't find a simple proof, so present here as an appendix.

**4.1. The Equivalence of Three Definitions for Gårding Cone.** The Gårding cone is originated from the Gårding theory of hyperbolic polynomials [2]. Each hyperbolic polynomial is essentially real. A homogeneous polynomial of  $k$ -order on a  $n$ -dimensional real vector space  $V$  is hyperbolic with respect to a direction  $a \in V$  if the equation  $P(sa + \lambda) = 0$  with one-variable  $s$  has  $k$  real zeros for every real  $\lambda \in V$ . Using a convenient solecism, we say  $P(\lambda)$  is an  $a$ -hyperbolic polynomial. For an  $a$ -hyperbolic polynomial  $P(\lambda)$  and  $m > 1$ , then by Rolle's theorem, its directional derivative in the direction  $a$ ,

$$(4.1) \quad (\nabla P(\lambda), a) = \frac{d}{ds} P(sa + \lambda) \big|_{s=0}$$

is also an  $a$ -hyperbolic polynomial, see Lemma 1 in [2]. For an  $a$ -hyperbolic polynomial  $P(\lambda)$  with  $P(a) > 0$ , its Gårding cone is defined by

$$(4.2) \quad \mathcal{C}(a, P, n) = \{\lambda \in \mathbb{R}^n : P(sa + \lambda) > 0, \forall s \geq 0\}.$$

Gårding [2] has proved that  $\mathcal{C}(a, P, n) = \mathcal{C}(b, P, n)$  is an open convex cone, for any  $b \in \mathcal{C}(a, P, n)$  with vertex at the origin, and

$$(4.3) \quad (\nabla P(\lambda), \mu) \geq k P^{\frac{k-1}{k}}(\lambda) P^{\frac{1}{k}}(\mu), \quad \forall \lambda, \mu \in \mathcal{C}(a, P, n),$$

see (11) in [2], which is the simplest version of the general Gårding inequality, Theorem 5 in [2]. We will apply the results above to the  $k$ -th elementary symmetric polynomial

$$\sigma_k(\lambda) = \sum_{1 \leq i_1 < \dots < i_k \leq n} \lambda_{i_1} \cdots \lambda_{i_k},$$

which is hyperbolic with respect to

$$e = (1, 1, \dots, 1) \in \mathbb{R}^n.$$

Gårding cone is equivalently defined in the form which is easier to be verified than (4.2)

$$(4.4) \quad \Gamma_k(n) = \{\lambda \in \mathbb{R}^n | \sigma_j(\lambda) > 0, \forall j = 1, 2, \dots, k\},$$

From this definition, it follows that

$$\bar{\Gamma}_n(n) \subset \dots \subset \bar{\Gamma}_k(n) \subset \dots \subset \bar{\Gamma}_1(n).$$

Notice that Maclaurin's inequalities

$$(4.5) \quad \left[ \frac{1}{\binom{n}{k}} \sigma_k(\lambda) \right]^{\frac{1}{k}} \leq \left[ \frac{1}{\binom{n}{l}} \sigma_l(\lambda) \right]^{\frac{1}{l}}$$

hold for  $1 \leq l \leq k$ ,  $\lambda \in \Gamma_k(n)$ ; (see Lemma 15.12 in [10]). Denoting

$$\lambda + \varepsilon = (\lambda_1 + \varepsilon, \lambda_2 + \varepsilon, \dots, \lambda_n + \varepsilon)$$

and applying the formula

$$\sigma_k(\lambda + \varepsilon) = \sum_{j=0}^k C(j, k, n) \varepsilon^j \sigma_{k-j}(\lambda), \quad C(j, k, n) = \frac{\binom{n}{k} \binom{k}{j}}{\binom{n}{k-j}}, \lambda \in \mathbb{R}^n, \varepsilon \in \mathbb{R}.$$

to  $\lambda + \varepsilon$ ; (see Section 5, [8]), we obtain

$$(4.6) \quad \lambda + \varepsilon \in \Gamma_k(n), \quad \forall \varepsilon > 0, \lambda \in \overline{\Gamma_k(n)}.$$

For any fixed  $t$ -tuple  $\{i_1, i_2, \dots, i_t\} \subset \{1, 2, \dots, n\}$ , we define

$$(4.7) \quad \sigma_{k; i_1, i_2, \dots, i_t}(\lambda) = \frac{\partial^t \sigma_{k+t}(\lambda)}{\partial \lambda_{i_1} \dots \partial \lambda_{i_t}}.$$

and then for any  $i \in \{1, 2, \dots, n\}$ ,

$$(4.8) \quad \sigma_k(\lambda) = \lambda_i \sigma_{k-1; i}(\lambda) + \sigma_{k; i}(\lambda), \quad \forall \lambda \in \mathbb{R}^n.$$

Moreover, one can verify by (4.8) that

$$\sigma_{k; i_1, i_2, \dots, i_t}(\lambda) = \sigma_k(\lambda)|_{\lambda_{i_1} = \dots = \lambda_{i_t} = 0}.$$

Ivovichkina [8] introduced a cone defined by

$$(4.9) \quad \widetilde{\Gamma}_k(n) = \{\lambda \in \mathbb{R}^n | \sigma_{k-l; i_1, \dots, i_l} > 0, l = 0, 1, \dots, k, \}$$

and has proved that, by her own notion of "stability set of  $k$ -Hessian operator", the cones  $\Gamma_k(n)$  and  $\widetilde{\Gamma}_k(n)$  coincide. We will take the original definition (4.2) as the starting point to prove that the three definitions above are equivalent.

**Theorem 4.1.** *For the  $k$ -th elementary symmetric polynomial  $\sigma_k(\lambda)$  with  $e = (1, 1, \dots, 1)$ , the definitions (4.2), (4.4) and (4.9) are equivalent, that is,*

$$\mathcal{C}(e, \sigma_k, n) = \Gamma_k(n) = \widetilde{\Gamma}_k(n).$$

*Proof. Step 1.* We will prove  $\mathcal{C}(e, \sigma_k, n) = \Gamma_k(n)$ . We first prove  $\mathcal{C}(e, \sigma_k, n) \subset \Gamma_k(n)$ . By definition of hyperbolic polynomial,

$$\sigma_n(\lambda) = \lambda_1 \lambda_2 \dots \lambda_n$$

is hyperbolic with respect to  $e = (1, 1, \dots, 1)$ . Noticing

$$\sigma_n(se + \lambda) = \sum_{j=0}^n s^{n-j} \sigma_j(\lambda),$$

by the convention  $\sigma_0(\lambda) = 1$ , and

$$\frac{d^{n-j}}{s^{n-j}} \sigma_n(se + \lambda) |_{s=0} = C(n, j) \sigma_j(\lambda).$$

By using (4.1) and Rolle's Theorem again, we see that  $\sigma_k(\lambda)$ ,  $1 \leq k \leq n$  are hyperbolic with respect to  $e = (1, 1, \dots, 1)$ . Accordingly, by definition (4.2) of  $\mathcal{C}(e, \sigma_j, n)$ , ( $1 \leq j \leq k$ ) and letting  $s = 0$ , we obtain

$$\sigma_j(\lambda) > 0, \quad 1 \leq j \leq k.$$

This completes the proof  $\mathcal{C}(e, \sigma_k, n) \subset \Gamma_k(n)$ . Conversely, if  $\lambda \in \Gamma_k(n)$ , since

$$\sigma_k(se + \lambda) = \sum_{j=0}^k C(j, k, n) s^j \sigma_{k-j}(\lambda), \quad C(j, k, n) = \frac{\binom{n}{k} \binom{k}{j}}{\binom{n}{k-j}}$$

and by definition of  $\Gamma_k(n)$ ,

$$\sigma_{k-j}(\lambda) > 0, \quad 0 \leq j \leq k,$$

we see that  $\sigma_k(se + \lambda) > 0$  for all  $s \geq 0$  and therefore  $\lambda \in \mathcal{C}(e, \sigma_k, n)$ . This completes the proof of  $\Gamma_k(n) \subset \mathcal{C}(e, \sigma_k, n)$ .

**Step 2.** We will prove that  $\widetilde{\Gamma}_k(n) = \Gamma_k(n)$ . Obviously  $\widetilde{\Gamma}_k(n) \subset \Gamma_k(n)$ . It is left to prove  $\mathcal{C}(e, \sigma_k, n) \subset \widetilde{\Gamma}_k(n)$ . Let  $\lambda \in \mathcal{C}(e, \sigma_k, n)$ , we use (4.3) with  $P(\lambda) = \sigma_j(\lambda)(1 \leq j \leq k)$  and  $\mu = (1, 0, \dots, 0) = e_1$  and then obtain

$$(4.10) \quad \frac{\partial \sigma_j}{\partial \lambda_1}(\lambda) = (\nabla \sigma_j, e_1) \geq 0, \quad 1 \leq j \leq k.$$

We claim that

$$(4.11) \quad \frac{\partial \sigma_k}{\partial \lambda_1}(\lambda) > 0, \quad \forall \lambda \in \mathcal{C}(e, \sigma_k, n).$$

Otherwise, we assume that  $\frac{\partial \sigma_k}{\partial \lambda_1}(\lambda^0) = 0$  for some  $\lambda^0 \in \mathcal{C}(e, \sigma_k, n)$ . By (4.8),

$$\sigma_k(\lambda^0) = \lambda_1^0 \sigma_{k-1;1}(\lambda^0) + \sigma_{k;1}(\lambda^0) = \lambda_1^0 \frac{\partial \sigma_k}{\partial \lambda_1}(\lambda^0) + \sigma_{k;1}(\lambda^0) = \sigma_{k;1}(\lambda^0),$$

from which we obtain, by (4.4),

$$(4.12) \quad \sigma_{k-1;1}(\lambda^0) = \sigma_{k-1}(\lambda_2^0, \lambda_3^0, \dots, \lambda_n^0) = 0, 0 < \sigma_k(\lambda^0) = \sigma_{k;1}(\lambda^0) = \sigma_k(\lambda_2^0, \lambda_3^0, \dots, \lambda_n^0).$$

Moreover, by virtue of (4.10), we obtain

$$0 \leq \frac{\partial \sigma_j}{\partial \lambda_1}(\lambda^0) = \sigma_{j-1}(\lambda_2^0, \lambda_3^0, \dots, \lambda_n^0), \quad 1 \leq j \leq k.$$

Therefore we have proved that  $(\lambda_2^0, \lambda_3^0, \dots, \lambda_n^0) \in \widetilde{\Gamma}_k(n-1)$ . By (4.6), for  $\varepsilon > 0$ ,

$$(\lambda_2^0 + \varepsilon, \lambda_3^0 + \varepsilon, \dots, \lambda_n^0 + \varepsilon) \in \Gamma_k(n-1),$$

to which we apply the Maclaurin's inequalities (4.5) and obtain

$$\left[ \frac{1}{\binom{n-1}{k}} \sigma_k(\lambda_2^0 + \varepsilon, \lambda_3^0 + \varepsilon, \dots, \lambda_n^0 + \varepsilon) \right]^{\frac{1}{k}} \leq \left[ \frac{1}{\binom{n-1}{l}} \sigma_l(\lambda_2^0 + \varepsilon, \lambda_3^0 + \varepsilon, \dots, \lambda_n^0 + \varepsilon) \right]^{\frac{1}{l}}.$$

Letting  $\varepsilon \rightarrow 0^+$ , we have

$$\left[ \frac{1}{\binom{n-1}{k}} \sigma_k(\lambda_2^0, \lambda_3^0, \dots, \lambda_n^0) \right]^{\frac{1}{k}} \leq \left[ \frac{1}{\binom{n-1}{l}} \sigma_l(\lambda_2^0, \lambda_3^0, \dots, \lambda_n^0) \right]^{\frac{1}{l}},$$

which contradicts with (4.12) in case  $l = k-1$ , thus the claim (4.11) is true. Notice that

$$\frac{\partial \sigma_k}{\partial \lambda_1}(\lambda) = \sigma_{k-1}(\lambda_2, \lambda_3, \dots, \lambda_n),$$

then applying the Maclaurin's inequality to (4.10)-(4.11) leads to

$$(\lambda_2, \lambda_3, \dots, \lambda_n) \in \Gamma_{k-1}(n-1).$$

Now we can regard that  $(\lambda_2, \lambda_3, \dots, \lambda_n)$  is in the same position of  $(\lambda_1, \lambda_2, \dots, \lambda_n)$  as above, by an induction on  $k$  we can prove

$$\sigma_{k-l; i_1, \dots, i_l}(\lambda) > 0, \quad l = 0, 1, \dots, k.$$

That is  $\Gamma_k(n) \subset \widetilde{\Gamma}_k(n)$ . □

For the Gårding cone in the space of symmetric matrices, similar results to (4.11) can be seen Section 3 in [9].

The following is some by-product of (4.9). Assume that  $\lambda \in \Gamma_k(n)$  is in descending order,

$$\lambda_1 \geq \cdots \lambda_{p-1} \geq \lambda_p > 0 \geq \lambda_{p+1} \geq \cdots \lambda_n,$$

then

$$(4.13) \quad \begin{cases} p \geq k \\ 0 < \sigma_{k-1;1}(\lambda) \leq \sigma_{k-1;2}(\lambda) \leq \cdots \leq \sigma_{k-1,n}(\lambda). \end{cases}$$

Otherwise, if  $p < k$ , we have

$$\sigma_{1;\lambda_1,\lambda_2,\dots,\lambda_{k-1}} = \sigma_1(\lambda_k, \lambda_{k+1}, \dots, \lambda_n) = \sum_{j=k}^n \lambda_j \leq 0,$$

which contradicts with (4.9). Using (4.9) again, we have  $\sigma_{k-2;12}(\lambda) \geq 0$  and then

$$\begin{aligned} \frac{\partial \sigma_k}{\partial \lambda_1}(\lambda) &= \sigma_{k-1;1}(\lambda) = \sigma_{k-1;12}(\lambda) + \lambda_2 \sigma_{k-2;12}(\lambda) \\ &\leq \sigma_{k-1;12}(\lambda) + \lambda_1 \sigma_{k-2;12}(\lambda) = \sigma_{k-1;2}(\lambda) = \frac{\partial \sigma_k}{\partial \lambda_2}(\lambda), \end{aligned}$$

the remaining part of (4.13) can be proved similarly. Here the proof of (4.13) is adapted from [13].

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